

## $\Gamma$ -TM-Algebra

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### Abstract

In this paper a new notion called  $\Gamma$ -TM-algebra has been introduced. Definition and examples of  $\Gamma$ -TM-algebra are provided. Some results in TM-algebra are extended to  $\Gamma$ -TM-algebra with the new notion. It is proved that every  $\Gamma$ -TM-algebra  $X$  satisfying  $xyz = z$  is a trivial algebra. Let  $X$  be a  $\Gamma$ -TM-algebra. Then  $x\gamma y = x\beta y$  for any  $x, y \in X$  and  $\beta, \gamma \in \Gamma$  which plays a major role has been proved.

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### Keywords

Binary relation, TM-algebra, trivial algebra.

### 1.Introduction

A class of Abstract Algebra known as **BCK-algebra** was introduced by Imai and Iseki (1966). In later another class of algebra called **BCI-algebra** was introduced and these two classes act as logical algebras. The notion **d-algebra** is the generalization of BCK-algebra was introduced by J.Negger and Kim (1999). A generalization of BCH/BCI/ BCK-algebra was introduced by Jun, Roh and Kim (1998) in the notion called **BH-algebra**. **Q-algebras** which is generalization of BCH/BCI/BCK-algebras was introduced by Ahn, Neggars and Kim (2001). A class of abstract algebras: **TM-algebras**, which is a generalization of Q/BCK/BCI/BCH-algebras was introduced by K.Megalai and A. Tamilarasi. Some results in TM-algebra are extended to  $\Gamma$ -TM-algebra with the new notion.

### 2.Preliminaries

#### Definition 2.1 [2]

A **BCK-algebra** is an algebra  $(X, *, 0)$  of type  $(2,0)$  satisfying the following conditions:

- i)  $(x * y) * (x * z) \leq z * y$
- ii)  $x * (x * y) \leq y$
- iii)  $x \leq x$
- iv)  $x \leq y$  and  $y \leq x$  imply  $x = y$
- v)  $0 \leq x$  implies  $x = 0$ , where  $x \leq y$  is defined by  $x * y = 0$  for all  $x, y, z \in X$ .

In any BCK-algebra  $(X, *, 0)$ , the following holds good for all  $x, y, z \in X$ .

- a)  $x * 0 = 0$
- b)  $x * y = 0$  and  $y * z = 0$  imply  $x * z = 0$
- c)  $x * y = 0$  implies  $(x * z) * (y * z) = 0$  and  $(z * y) * (z * x) = 0$
- d)  $(x * y) * z = (x * z) * y$
- e)  $(x * y) * x = 0$
- f)  $x * (x * (x * y)) = x * y$

- g)  $(x * y) * z = 0$  implies  $(x * z) * y = 0$
- h)  $((x * y) * (y * z)) * (x * y) = 0$
- i)  $((x * z) * z) * (y * z) * ((x * y) * z) = 0$
- j)  $(x * z) * (x * (x * z)) = (x * z) * z$
- k)  $\left( x * (y * (y * x)) * \left( y * \left( x * (y * (y * x)) \right) \right) \right) * (x * y) = 0$

**Definition 2.2 [2]**

A **BCI-algebra** is an algebra  $(X, *, 0)$  of type  $(2,0)$  satisfying the following conditions:

- i)  $(x * y) * (x * z) \leq z * y$
- ii)  $x * (x * y) \leq y$
- iii)  $x \leq x$
- iv)  $x \leq y$  and  $y \leq x$  imply  $x = y$
- v)  $x \leq 0$  implies  $x = 0$ , where  $x \leq y$  is defined by  $x * y = 0$  for all  $x, y, z \in X$ .

In a BCI-algebra  $X$ , the following hold:

- a)  $(x * y) * z = (x * z) * y$
- b)  $x * 0 = x$
- c)  $x \leq y$  implies  $x * z \leq y * z$  and  $z * y \leq z * x$
- d)  $(x * z) * (y * z) \leq x * y$
- e)  $x * (x * (x * y)) = x * y$

**Definition 2.3 [3]**

A **d-algebra** is a non-empty set  $X$  with a constant  $0$  and a binary operation  $*$  satisfying the following axioms:

- i)  $x * x = 0$
- ii)  $0 * x = 0$
- iii)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$  for all  $x, y \in X$ .

**Definition 2.4 [4]**

A **BH-algebra** is a non-empty set  $X$  with a constant  $0$  and a binary operation  $*$  satisfying the following axioms:

- i)  $x * x = 0$
- ii)  $x * 0 = x$
- iii)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$  for all  $x, y \in X$ .

**Definition 2.5 [2]**

A **Q-algebra** is a non-empty set  $X$  with a constant  $0$  and a binary operation  $*$  satisfying the axioms:

- i)  $x * x = 0$
- ii)  $x * 0 = x$
- iii)  $(x * y) * z = (x * z) * y$  for all  $x, y, z \in X$ .

Every Q-algebra  $X$  satisfies the following condition:

- iv)  $(x * (x * y)) * y = 0$  for any  $x, y \in X$ .

**Definition 2.6 [1]**

A **TM-algebra**  $(X, *, 0)$  is a non-empty set  $X$  with a constant  $0$  and a binary operation  $*$  satisfying the following axioms:

- i)  $x * 0 = x$  for  $x \in X$
- ii)  $(x * y) * (x * z) = z * y$  for  $x, y, z \in X$ .

In  $X$  we can define a binary relation  $\leq$  by  $x \leq y$  if and only if  $x * y = 0$ .

**3.  $\Gamma$ -TM-algebra**

**Definition 3.1:**

Let  $X$  and  $\Gamma$  be any non-empty sets. If there exists a mapping  $X \times \Gamma \times X \rightarrow X$  written as  $(x, \gamma, y)$  by  $x\gamma y$ , then  $(\Gamma, X, 0)$  is called a  **$\Gamma$ -TM-algebra** if

- i)  $x\gamma 0 = x$  for all  $x \in X$  and  $\gamma \in \Gamma$
- ii)  $(x\gamma y)\beta(x\alpha z) = z\beta y$  for all  $x, y, z \in X$  and  $\alpha, \beta, \gamma \in \Gamma$ .

In a  $\Gamma$ -TM-algebra  $X$  we can define a binary relation  $\leq$  by  $x \leq y$  if and only if  $x\gamma y = 0$  for all  $\gamma \in \Gamma$  and  $x, y \in X$ .

**Example 3.2:**

Let  $(\Gamma, X, 0)$  be an arbitrary TM-algebra and  $\Gamma$  any non-empty set. Define a mapping  $X \times \Gamma \times X \rightarrow X$  by  $x\gamma y = xy$  for all  $x, y \in X$  and  $\gamma \in \Gamma$ .

It is easy to see that  $X$  is a  $\Gamma$ -TM-algebra.

Indeed,

- i)  $x\gamma 0 = x0 = x$
- ii)  $(x\gamma y)\beta(x\alpha z) = (xy)\beta(xz)$   
 $= (xy)(xz)$   
 $= zy$   
 $= z\beta y$  for all  $x, y, z \in X$  and  $\alpha, \beta, \gamma \in \Gamma$ .

Thus every TM-algebra implies a  $\Gamma$ -TM-algebra.

**Proposition 3.3:**

If  $(\Gamma, X, 0)$  is a  $\Gamma$ -TM-algebra, then

- i)  $x\beta x = 0$
- ii)  $(x\gamma y)\beta x = 0\beta y$
- iii)  $x\alpha(x\beta y) = y$  for any  $x, y \in X$  and  $\alpha, \beta, \gamma \in \Gamma$ .

**Proof:**

Let  $(\Gamma, X, 0)$  be a  $\Gamma$ -TM-algebra.

Let  $x, y \in X$  and  $\alpha, \beta, \gamma \in \Gamma$ .

- i)  $x\beta x = (x\gamma 0)\beta(x\alpha 0)$  (by definition 3.1(i))  
 $= 0\beta 0$  (by definition 3.1(ii))  
 $= 0$

Hence  $x\beta x = 0$

- ii)  $(x\gamma y)\beta x = (x\gamma y)\beta(x\alpha 0)$  (by definition 3.1(i))  
 $= 0\beta y$  (by definition 3.1(ii))

Hence  $(x\gamma y)\beta x = 0\beta y$

- iii)  $x\alpha(x\beta y) = (x\gamma 0)\alpha(x\beta y)$  (by definition 3.1(i))  
 $= y\alpha 0$  (by definition 3.1(ii))

$= y$  (by definition 3.1(i))  
Hence  $x\alpha(x\beta y) = y$

**Proposition 3.4:**

Let  $(\Gamma, X, 0)$  be a  $\Gamma$ -TM-algebra. Then  $(x\gamma y)\beta z = (x\gamma z)\beta y$  for any  $x, y, z \in X$  and  $\beta, \gamma \in \Gamma$ .

**Proof:**

Let  $X$  be a  $\Gamma$ -TM-algebra.

Let  $x, y, z \in X$  and  $\beta, \gamma \in \Gamma$ .

Then by definition 3.1(ii), we have  $(x\gamma y)\beta(x\alpha z) = z\beta y$  (3.1)

Put  $z = x\gamma y$  and  $y = z$  in equation (3.1).

Then  $(x\gamma y)\beta z = (x\gamma z)\beta(x\alpha(x\gamma y))$   
 $= (x\gamma z)\beta y$  (by Proposition 3.3 (iii))

Hence  $(x\gamma y)\beta z = (x\gamma z)\beta y$ .

**Lemma 3.5:**

Let  $X$  be a  $\Gamma$ -TM-algebra. Then  $x\gamma y = x\beta y$  for any  $x, y \in X$  and  $\beta, \gamma \in \Gamma$ .

**Proof:**

Let  $X$  be a  $\Gamma$ -TM-algebra.

Let  $x, y \in X$  and  $\beta, \gamma \in \Gamma$ .

Then  $x\gamma y = (x\beta 0)\gamma y$  (by definition 3.1(i))  
 $= (x\beta y)\gamma 0$  (by Proposition 3.4)  
 $= x\beta y$  (by definition 3.1(i))

Hence  $x\gamma y = x\beta y$ .

**Proposition 3.6:**

Any  $\Gamma$ -TM-algebra  $X$  satisfies for any  $x, y \in X$  and  $\alpha, \beta, \gamma \in \Gamma$ ,  $(y\beta(y\alpha x))\gamma x = 0$ .

**Proof:**

Let  $X$  be a  $\Gamma$ -TM-algebra.

Let  $x, y \in X$  and  $\alpha, \beta, \gamma \in \Gamma$ .

Then  $(y\beta(y\alpha x))\gamma x = (y\beta x)\gamma(y\alpha x)$  (by Proposition 3.4)  
 $= (y\alpha x)\gamma(y\alpha x)$  (by lemma 3.5)  
 $= 0$  (by Proposition 3.3(i))

Hence  $(y\beta(y\alpha x))\gamma x = 0$ .

**Proposition 3.7:**

Let  $(\Gamma, X, 0)$  be a  $\Gamma$ -TM-algebra. Then for any  $x, y, z \in X$  and  $\alpha, \beta, \gamma, \lambda, \theta \in \Gamma$

- a)  $x\gamma 0 = 0$  implies  $x = 0$
- b)  $(x\gamma z)\beta(y\alpha z) \leq x\beta y$
- c)  $x\lambda(x\alpha(x\beta y)) = x\beta y$
- d)  $(0\gamma x)\lambda(0\gamma y) = 0\lambda(x\gamma y)$
- e)  $(x\gamma(x\beta y))\alpha y = 0$
- f) If  $x\gamma y = 0, y\gamma x = 0$  then  $x = y$ .

**Proof:**

a)  $x\gamma 0 = x$  (by definition 3.1 (i))

If  $x\gamma 0 = 0$ , then  $x = 0$ , proves the result.

b) To prove  $(x\gamma z)\beta(y\alpha z) \leq x\beta y$

That is, to prove  $((x\gamma z)\beta(y\alpha z))\lambda(x\beta y) = 0$

$$\begin{aligned} ((x\gamma z)\beta(y\alpha z))\lambda(x\beta y) &= ((x\gamma z)\beta(x\beta y))\lambda(y\alpha z) \text{ (by Proposition 3.4)} \\ &= ((x\gamma z)\beta(x\theta y))\lambda(y\alpha z) \text{ (by lemma 3.5)} \\ &= (y\beta z)\lambda(y\alpha z) \text{ (by definition 3.1(ii))} \\ &= (y\beta z)\lambda(y\beta z) \text{ (by lemma 3.5)} \\ &= 0 \text{ (by Proposition 3.3(i))} \end{aligned}$$

Thus  $((x\gamma z)\beta(y\alpha z))\lambda(x\beta y) = 0$

Hence  $(x\gamma z)\beta(y\alpha z) \leq x\beta y$

c)  $x\lambda(x\alpha(x\beta y)) = (x\gamma 0)\lambda(x\alpha(x\beta y))$  (by definition 3.1(i))  
 $= (x\beta y)\lambda 0$  (by definition 3.1(ii))  
 $= x\beta y$  (by definition 3.1(i))

Hence  $x\lambda(x\alpha(x\beta y)) = x\beta y$

d)  $0\lambda(x\gamma y) = (x\alpha x)\lambda(x\gamma y)$  (by Proposition 3.3(i))  
 $= y\lambda x$  (by definition 3.1(ii))  
 $= (0\gamma x)\lambda(0\beta y)$  (by definition 3.1(ii))  
 $= (0\gamma x)\lambda(0\gamma y)$  (by lemma 3.5)

Hence  $(0\gamma x)\lambda(0\gamma y) = 0\lambda(x\gamma y)$

e)  $(x\gamma(x\beta y))\alpha y = (x\gamma y)\alpha(x\beta y)$  (by Proposition 3.4)  
 $= (x\beta y)\alpha(x\beta y)$  (by lemma 3.5)  
 $= 0$  (by Proposition 3.3(i))

Hence  $(x\gamma(x\beta y))\alpha y = 0$

f)  $x = x\beta 0$   
 $= x\beta(x\gamma y)$   
 $= y$  (by Proposition 3.3(iii))

Hence  $x = y$ .

(or)

$y = y\beta 0$   
 $= y\beta(y\gamma x)$   
 $= x$  (by Proposition 3.3(iii))

Hence  $x = y$ .

**Proposition 3.8:**

Let  $X$  be a  $\Gamma$ -TM-algebra. If  $x\gamma(x\beta y) = x\beta y$  for any  $x, y \in X$  and  $\beta, \gamma \in \Gamma$ , then  $x\gamma 0 = 0$ .

**Proof:**

Let  $X$  be a  $\Gamma$ -TM-algebra.

Let  $x \in X$  and  $\beta, \gamma \in \Gamma$ .

Given  $x\gamma(x\beta y) = x\beta y$  (3.2)

Then  $x\gamma 0 = x\gamma(x\beta x)$  (by Proposition 3.3(i))  
 $= x\beta x$  (by equation (3.2))

$$= 0$$

Hence  $x\gamma 0 = 0$ .

**Theorem 3.9:**

Every  $\Gamma$ -TM-algebra  $X$  satisfying  $x\gamma z = z$  is a trivial algebra.

**Proof:**

Let  $X$  be a  $\Gamma$ -TM-algebra.

Let  $x, z \in X$  and  $\gamma \in \Gamma$ .

Given  $x\gamma z = z$  (3.3)

Put  $x = z$  in equation (3.3)

Then  $z\gamma z = z$

$$\Rightarrow 0 = z \text{ (by Proposition 3.3(i))}$$

Hence  $X$  is a trivial algebra.

**Lemma 3.10:**

Let  $X$  be a  $\Gamma$ -TM-algebra. Then  $a\gamma b = a\gamma c$  for all  $a, b, c \in X$  and  $\gamma \in \Gamma$  implies  $0\gamma b = 0\gamma c$ .

**Proof:**

Let  $X$  be a  $\Gamma$ -TM-algebra.

Let  $a, b, c \in X$  and  $\beta, \gamma \in \Gamma$ .

By Proposition 3.4 we have,  $(x\gamma y)\beta z = (x\gamma z)\beta y$  (3.4)

Put  $x = z = a$  and  $y = b$  in equation (3.4)

We have  $(a\gamma b)\beta a = (a\gamma a)\beta b = 0\beta b$  (by Proposition 3.3(i))

Put  $x = z = a$  and  $y = c$  in equation (3.4)

We have  $(a\gamma c)\beta a = (a\gamma a)\beta c = 0\beta c$  (by Proposition 3.3(i))

Therefore  $a\gamma b = a\gamma c$

$$\Rightarrow (a\gamma b)\beta a = (a\gamma c)\beta a$$

$$\Rightarrow 0\beta b = 0\beta c$$

$$\Rightarrow 0\gamma b = 0\gamma c \text{ (by lemma 3.5)}$$

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